Exact solution of return hysteresis loops in a one-dimensional random-field Ising model at zero temperature

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Minor hysteresis loops within the main loop are obtained exactly in the one-dimensional ferromagnetic random-field Ising model at zero temperature. Numerical simulations of the model show excellent agreement with the exact results.

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I. INTRODUCTION

Hysteresis is observed in any material that is driven by a force cycling faster than it can equilibrate. It has practical importance and old scientific interest [1] renewed by the present focus of statistical mechanics on nonequilibrium phenomena. There have been many theoretical studies of hysteresis recently, and also simulations and experiments [2–5]. The purpose of the present paper is to make a small contribution in the context of exactly solvable models of hysteresis. Work on exactly solvable models of hysteresis forms a very small fraction of the total work on hysteresis, but it goes back to the earliest attempts at a theory of hysteresis. Rayleigh, Preisach, and Stoner-Wohlfarth [1] studied models of hysteresis that could be solved exactly. However, these models neglected the frequency dependence of hysteresis loops and were purely phenomenological. For example, in the Preisach model, a ferromagnet is assumed to consist of many independent magnetic domains of varying sizes. Each domain is assumed to have a rectangular hysteresis loop characterized by two coercive fields. Each domain is also assumed to relax instantaneously, and therefore the hysteresis loops have no frequency dependence. The distribution of the size of domains and their coercive fields is varied in order to fit experimental hysteresis loops. The Preisach model is particularly successful in fitting to experimental results, but it does not attempt to explain why the individual domains have rectangular loops. Subsequently, several models based on the dissipative Langevin dynamics of Heisenberg ferromagnets in the presence of an oscillating external field have been studied, and exact results have been obtained for the frequency dependence of the hysteresis loops in some special cases [3]. More recently, Sethna *et al.* [4] have studied the zero temperature dynamics of the ferromagnetic random-field Ising model (RFIM) on a lattice as a model of hysteresis and Barkhausen noise in ferromagnets. The limitation to the zero temperature dynamics means that the model of Sethna *et al.* neglects the frequency dependence of the hysteresis loops just like the earliest models of hysteresis mentioned above, but it has the advantage of being a microscopic model and describes several other phenomena as well [6], including athermal martensitic transformations, fluid flow in porous media, and pinning of flux lines in superconductors. The zero temperature dynamics of the ferromagnetic RFIM has been solved exactly in the mean field approximation [4], as well as in one dimension and on a Bethe lattice [7,8] in the case when the system evolves from a saturated state. The antiferromagnetic RFIM has also been solved exactly in one dimension [9] if the system evolves from an initial state with all spins parallel to each other. The restriction to an initial state with all spins parallel to each other means that solutions for minor hysteresis loops lying within the main hysteresis loop could not be obtained even in one dimension so far. In the present paper we are able to lift this restriction for the ferromagnetic RFIM in one dimension. We present exact solutions of return hysteresis loops starting anywhere on the parent loop.

II. STARTING WITH A SATURATED STATE

The one-dimensional random-field Ising model is characterized by the Hamiltonian

$$H=-J\sum_i s_i s_{i+1}-\sum_i h_i s_i-h\sum_i s_i\,. \qquad (1)$$

Here $s_i=\pm 1$ are the Ising spins, h_i is the quenched random field drawn from a continuous probability distribution $p(h_i)$, and h is the external field. The zero temperature dynamics amounts to flipping a spin only if it lowers the energy of the system. It normally causes an avalanche, i.e., a large number of neighboring spins have to be flipped before the system comes to a stable state. We keep the applied field fixed during an avalanche, and raise it afterwards until the next avalanche occurs. More details of the model can be found in Refs. [4,7,8]. The ferromagnetic RFIM ($J\geqslant 0$) has two important properties. It is Abelian [8], i.e., the stable state after an avalanche does not depend upon the order in which the spins flip during an avalanche. And it has return point memory [4], i.e., the stable state in a slowly changing field h depends only on the state where this field was last reversed. In the special case when we start at $h=-\infty$ and raise the field monotonically, the state at h does not depend on the rate of increase in h. Large rates of increase result in fewer but larger avalanches, and small rates in more numerous but smaller avalanches. The final state remains the same. We exploit this property in determining the stable state at h through a single large avalanche from the initial state at $h=-\infty$. The Abelian property tells us that, during this avalanche, whether a spin at site i flips or not depends on the quenched field h_i on the site and the number of nearest

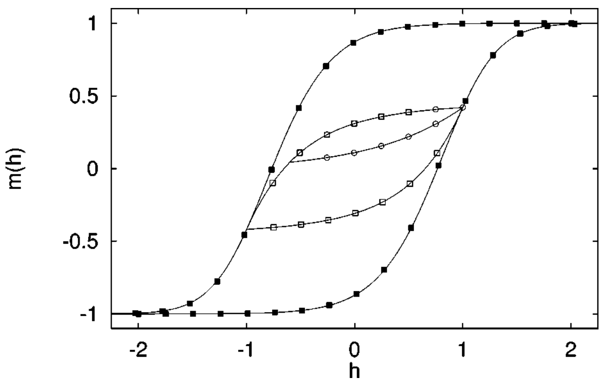

FIG. 1. Hysteresis loop (filled squares) between two saturated states for a Gaussian random field (mean=0, variance=1, $J=1$). Two excursions from the lower half are shown: $h=1$ to $h'=-1$ and back (open squares), and $h=1$ to $h'=-0.6$ and back (open circles).

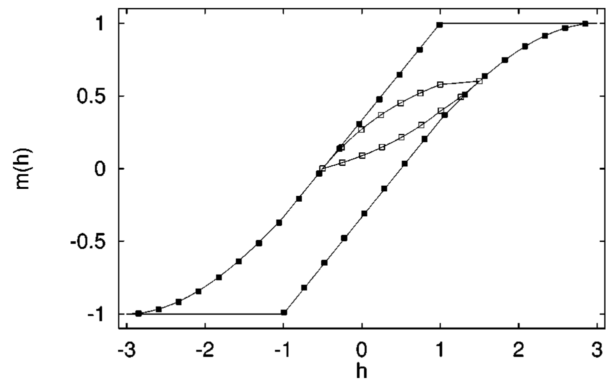

FIG. 2. Hysteresis loop (filled squares) for a rectangular distribution of the random field of width 6 ($J=1$). Return loop (open squares) shows an excursion from the lower half ($h=1.5$ to $h'=-0.5$ and back).

neighbors n ($n=0,1,2$) that have flipped up before it, but not on the order in which the neighbors flipped. This probability is given by

$$p_n(h)=\text{prob}[h_i+2(n-1)J+h]\geqslant 0=\int_{2(1-n)J-h}^{\infty} p(h_i)dh_i .$$

We now need to calculate the probability that a nearest neighbor of a site i flips up before site i. Let us denote the conditional probability that site $i+1$ (or site $i-1$) flips up before site i by $P^*(h)$. There are many ways in which the site $i+1$ could be up, and we must sum over all the possibilities to calculate $P^*(h)$. If site i is down and site $i+1$ is up, a spin at site $i+m$ ($m\geqslant 1$) must have flipped up before any of its neighbors were up with probability $p_0(h)$, and then the spins from $i+m$ to $i+1$ must have flipped up in an avalanche. Summing over these cases, we get

$$P^*(h)=\sum_{m=1}^{\infty} p_0(h)[p_1(h)-p_0(h)]^{m-1}$$

or

$$P^*(h)=\frac{p_0(h)}{1-[p_1(h)-p_0(h)]}.$$

The probability that an arbitrary site is up at field h is given by

$$p(h)=[P^*(h)]^2p_2(h)+2P^*(h)[1-P^*(h)]p_1(h) +[1-P^*(h)]^2p_0(h). \tag{2}$$

The magnetization per spin $m(h)$ is related to $p(h)$ by the simple equation $m(h)=2p(h)-1$. The lower half of the large hysteresis loop in Fig. 1 shows $m(h)$ for a Gaussian distribution of the quenched field, and in Fig. 2 it is shown for a rectangular distribution. The upper half of the main loop in each case has been obtained by symmetry, $m_u(h)=-m(-h)$.

III. REVERSING THE APPLIED FIELD

Reversing the applied field from $h=+\infty$ does not constitute a new problem because the upper half of the large hysteresis loop shown in Fig. 1 can be obtained from the lower half by symmetry. However, reversing the applied field from any other point constitutes a new and somewhat more difficult problem. The reason is that, in a starting state at a finite field h, whether the spin at a site is flipped or not depends in a nontrivial way on the random field at that site as well as on neighboring sites. The state is thus ''strongly correlated,'' and it is difficult to do perturbation theory about this state.

The calculation of the return trajectory is rather tedious. A few preliminary remarks may be helpful to the reader before taking up the calculation. Consider a site i that is up at the point of return on the lower hysteresis loop and turns down on the return trajectory. Let h_i be the quenched field on the site i, h the applied field at the point of return, and h' the applied field when site i turns down. We characterize the site i by a set of three integers n, n', and n'', each of which can take three possible values 0, 1, or 2. The reason why we need three integers to characterize the site i will become clearer in due course. Let us first define what the three integers represent: n is the number of nearest neighbors of site i that are up just before site i flips up on the lower hysteresis loop; n' is the number of nearest neighbors of site i that are up at the start of the return trajectory at applied field h ($n'\geqslant n$); n'' is the number of nearest neighbors of site i that are up just before it flips down at applied field h' ($n''\leqslant n'$). The probability for the site i to turn down at h' obviously depends on h_i, h', and n''. If the site i initiates a downward avalanche, i.e., if it turns down before any of its nearest neighbors turn down, then $n''=n'$. If the site i turns down during an avalanche, i.e., if it turns down after one or more of its nearest neighbors have turned down, then $n''<n'$. The importance of the integer n is that it characterizes the relative strength of the quenched field h_i at the site i. For example, if $n=0$, we must have $h_i-2J+h\geqslant 0$, i.e., the quenched field on the site i must be relatively large ($h_i\geqslant 2J-h$). The *a posteriori* distribution of the quenched field on the up spins at the point of return is strongly modified from the initial distribution. We classify the up spins at the point of return into three main categories characterized by the value of the integer n, and

each of these categories into subcategories characterized by the values of n' and n''.

We start backtracking from an arbitrary applied field h on the lower loop, and come down to h' $(h' \leqslant h)$. We want the magnetization at h'. Obviously, spins can flip down only on the reverse trajectory, and therefore we focus on spins that are up at h but turn down at h'. As discussed above, we divide the up spins at h into three main categories characterizing the range of their random field and how they turned up on the lower hysteresis loop. Spins in category 0 have $h_i \geqslant 2J-h$. These spins could turn up at h even if none of their neighbors were up to help them. Spins in category 1 have $-h \leqslant h_i \leqslant 2J-h$, and spins in category 2 have $-2J-h \leqslant h_i \leqslant -h$. No spin could be up at h if it has $h_i \leqslant -2J-h$. The three main categories listed above are determined by the number of up neighbors n that a spin has just before it turns up during an upward avalanche at h. After that avalanche has settled, the number of up neighbors may increase. Thus each of the three main categories can be further divided into three subcategories characterized by the number of up neighbors n' after the avalanche. Some of the subcategories may be empty because $n' \geqslant n$. For example, if $n=2$, there are no subcategories with $n'=0$, or 1. A spin of category 2 $(n=2)$ that is up at h necessarily has both neighbors up $(n'=2)$. Spins of category 1 could have one or both neighbors up. Spins of category 0 could have zero, one, or two neighbors up at h. An important point to note is that, when the applied field is reversed, spins of category 2 with both neighbors up are as susceptible to turning down as spins of category 1 with one neighbor up because the net field in both cases lies in the same range.

In the first instance, we consider a restricted range of the reversed field: $h-2J \leqslant h' \leqslant h$. In this range, the only spins that could turn down are spins of category 2 with two neighbors up, spins of category 1 with one neighbor up, and spins of category 0 with zero neighbors up. We add the contributions from these three categories, and subtract the sum from the number of up spins at h. This gives us the magnetization at h'. Consider the spins of category 2 first: their fraction at h is equal to $[P^*(h)]^2[p_2(h)-p_1(h)]$. The factor $[P^*(h)]$ gives the probability that a nearest neighbor of a spin is up on the lower hysteresis loop before that spin is relaxed. Thus $[P^*(h)]^2$ is the probability that both neighbors of the spin are up before it is relaxed. The factor $[p_2(h)-p_1(h)]$ gives the probability that the spin turns up if two neighbors are up but not if only one neighbor is up. Thus, the fraction of category 2 spins that turn down at h' on the return loop is given by

$$q_r^2(h,h') = [P^*(h)]^2[p_2(h)-p_2(h')].$$

Now we take up the spins of category 1. In the initial state at h, category 1 spins come in two subcategories: (i) spins with one neighbor up and (ii) spins with two neighbors up. In the restricted range of the reversed field $(h-2J \leqslant h' \leqslant h)$, spins in subcategory (ii) cannot turn down spontaneously. However, they can turn down in an avalanche, if the avalanche puts one of their neighbors in category (ii) and it turns down. An avalanche can start with a category 1 spin that has one neighbor down in the starting state at h. This occurs with the probability $f(h)$ given by

$$f(h) = \{1-p_2(h)\}[P^*(h)] + \{1-p_1(h)\}[1-P^*(h)].$$

The above equation can be understood as follows. Suppose the spin at site i is up, and $f(h)$ denotes the probability that the spin at site $i+1$ is down. Before the spin at site $i+1$ is relaxed, the spin at site $i+2$ is up with the probability $[P^*(h)]$ and down with the probability $[1-P^*(h)]$. The probability that the spin stays down in the two cases even after it is relaxed is given by $\{1-p_2(h)\}$ and $\{1-p_1(h)\}$, respectively. The probability for the spin at $i+1$ to flip down at h' is equal to $[p_1(h)-p_1(h')]$. After it flips down, the spin at $i-1$ can also flip down with the same probability if it belongs to category 1 and the spin at $i-2$ is up. Thus an avalanche can start. The avalanche will go on until it meets a category-1 spin that does not flip down at h' or it meets a category 0 spin that has an up neighbor on the other side. The probability that a nearest neighbor of an up spin is down at h' is given by,

$$q_a(h,h') = \frac{f(h)}{1-[p_1(h)-p_1(h')]}.$$

Here, $f(h)$ is the probability that the neighbor was already down in the initial state. The other factor is the sum of an infinite series that accounts for avalanches of various sizes which may bring the neighbor down.

An avalanche can also be started by a spin of category 2 flipping down. This gives another term,

$$q_b(h,h') = \frac{[p_2(h)-p_2(h')][P^*(h)]}{1-[p_1(h)-p_1(h')]}.$$

The numerator in the above equation can be understood as follows. Suppose the spins at sites i, $i+1$, and $i+2$ are up and site $i+1$ belongs to category 2. $[P^*(h)]$ is the probability that site $i+2$ was up before site $i+1$ was relaxed at h. The numerator gives the probability that the right side neighbor of the up spin at site i flips down at h'. The denominator takes care of any possible avalanches started by the flipping down of a category 2 site. The total probability that a nearest neighbor of an up spin is down at h' is equal to q_a+q_b. We also need the probability that a nearest neighbor of an up spin is up before that spin is relaxed at h'. This is equal to the probability that the neighbor in question was up on the lower hysteresis loop before the site was relaxed at h, i.e., it is equal to $P^*(h)$. With this knowledge, we are now in a position to write the fraction of category 1 spins that turn down on the return loop at h'. We get

$$q_r^1(h,h') = 2[P^*(h)][q^a(h,h')+q_b(h,h')][p_1(h) - p_1(h')].$$

Spins of category 1 cannot have both nearest neighbors down. The reason is that this class of spins are flipped up during an avalanche on the lower hysteresis loop. Therefore they must be connected by up spins to a spin of category 0 on one side at least. A spin of category 0 cannot turn down if it has at least one neighbor up. However, if both neighbors of a spin of category 0 are down at h', it may turn down. The fraction of such spins is given by,

$$q_r^0(h,h')=[q^a(h,h')+q_b(h,h')]^2[p_0(h)-p_0(h')].$$

We are now in a position to write the magnetization on the return loop in the range $[h-2J\leqslant h'\leqslant h]$. We get, $m'(h')=2p'(h')-1$, where

$$p'(h')=p(h)-q_r^2(h,h')-q_r^1(h,h')-q_r^0(h,h'). \quad (3)$$

The key to getting the return magnetization beyond the range considered above is to note that the state of the system on the reverse trajectory at $h'=h-2J$ is the same as would be obtained from the initial state at $h'=+\infty$. If the initial state at $h'=\infty$ is exposed to an applied field $h-2J$, spins with $h_i\leqslant -h$ will flip down spontaneously and start avalanches where the adjacent spins in the range $-h\leqslant h_i\leqslant 2J-h$ will flip down. When this avalanche is finished, the remaining up spins will belong to three categories: (i) spins with $h_i\geqslant 2J-h$ with one neighbor up, (ii) spins with $h_i\geqslant 4J-h$ with no neighbors up, and (iii) spins with $h_i\geqslant -h$ with two neighbors up. This is precisely the state obtained at the end of the reverse trajectory obtained above. Therefore, the reverse trajectory in the range $h'\leqslant h-2J$ merges with the upper half of the big hysteresis loop.

IV. REVERSING THE FIELD AGAIN

The magnetization in reversed field merges with the upper half of the big hysteresis loop when the field falls below $h-2J$. Pulling up the field from below $h-2J$ can be related by symmetry to the problem of the return loop analyzed in the previous section. We need not repeat this calculation. However, if the reversed field is re-reversed before it reaches $h-2J$, we have a new problem on our hands, which we now analyze.

We turn back the field at h'. Our object is to obtain the magnetization at an arbitrary value h'' $(h'\leqslant h''\leqslant h)$ on the lower half of the return loop. Essentially, we are looking at the same strings of spins that turned down in the previous section, but now they turn up from the other end. If a spin is down on the lower half of the return loop, it must have been down at the end of the upper half as well. The reason is that on the lower half spins can only flip up, none can flip down. Thus the probability that a nearest neighbor of a down spin is down on the lower return loop is equal to $q_a(h,h')+q_b(h,h')$. The probability that the nearest neighbor is up increases steadily as more spins flip up on the lower half. First, let us look at the probability of an up neighbor at the start of the lower return loop. Consider three adjacent sites: $i-1$, i, and $i+1$. Given that site $i+1$ is down, we want the probability that site i is up. It follows from the previous section that, if site i is up at h', it must be a spin of category 0, or there must be a string of up spins to the left of i containing a spin of category 0. Spins of category 0 are up with probability unity if they are adjacent to an up spin, otherwise they have to have a quenched field in excess of $4J-h$. Thus the probability that site i is up and is a spin of category 0 is equal to $[1-(q_a+q_b)]p_0(h)+(q_a+q_b)p_0(h'')$. The probability that site i is up and not a spin of category 0 is equal to $[P^*(h)][p_1(h')-p_0(h)]$. Putting it together, the probability that a nearest neighbor of a down spin is up on the lower return loop before that neighbor is relaxed is given by

$$p_{rr}(h,h',h'')=\frac{a}{1-[p_1(h'')-p_1(h')]},$$

where,

$$a=[p_1(h')-p_0(h)]P^*(h)+[1-(q_a+q_b)]p_0(h)+(q_a+q_b)p_0(h'').$$

The magnetization on the lower return loop is given by $m''(h'')=2p''(h'')-1$, where

$$\begin{aligned}p''(h'')={}&p'(h')+(q_a+q_b)^2[p_0(h'')-p_0(h')]\\&+2(q_a+q_b)p_{rr}(h,h',h'')[p_1(h'')-p_1(h')]\\&+p_{rr}^2(h,h',h'')[p_2(h'')-p_0(h')].\end{aligned} \quad (4)$$

As may be expected, the exact results obtained above agree quite well with numerical simulations of the model. Figure 1 shows a comparison for a Gaussian distribution of the random field, and Fig. 2 for a rectangular distribution. The exact results involve the probabilities $p_0(h)$, $p_1(h)$, and $p_2(h)$, which are integrals over the random-field distribution. For a rectangular distribution of the random-field these integrals are linear functions of the applied field. However, for a Gaussian distribution of the random field, the integrals over the random field distribution become error functions which have to be evaluated numerically. The exact expressions are shown by continuous lines. Simulations for a chain of 1000 spins (averaged over 1000 different realizations of the random-field distribution) are indistinguishable from the exact expressions, but these are shown by large symbols at sparse intervals for visual convenience.

V. CONCLUDING REMARKS

The nonequilibrium response of RFIM at zero temperature is related to experimentally measurable quantities in several diverse systems. It has been calculated exactly in one dimension using probabilistic methods, and checked against numerical simulations of the model. The probabilistic method used here and in earlier work [7–9] assumes the existence of a unique thermodynamic state which is self-averaging. The numerical simulations do not involve making any assumptions about the thermodynamic state, and explicitly average the results for reasonably large systems over different realizations of the random field distribution. The agreement between the theoretical expressions and the simulation results is an indication that the zero temperature dynamics brings the RFIM to a self-averaging thermodynamic state.

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